The Problem of Two Aces—C.E. Mungan, Spring 1998

Boas problem 16.4.8 in paraphrased form asks the following. You are dealt two cards from a shuffled deck. (i) What is the probability of getting two aces? (ii) If you know that one is an ace, what is the probability that the other is an ace? (iii) If you know that one is the ace of spades, what is the probability that the other is an ace? Eric Nelson gave me the gist of the answer presented below. Note that I have paraphrased the question to emphasize that the answer does not depend on subtle wording issues such as "at least one is an ace" which only appears in one of the questions in Boas.

Let us use the fact that the probability of getting what you want is the ratio of the number of ways of getting what you want to the number of ways of getting anything possible. Let us call the number of ways of getting what you want N_{win}; then $N_{win} + N_{lose}$ is the number of ways of getting anything possible, where N_{lose} is the number of ways of getting what you do not want.

We can eliminate another distractor. Namely, you get the same answer regardless of whether you keep track of which card you were dealt first and which second, or whether you say you have a group of two cards independent of order of dealing. In the first case, N_{win} and N_{lose} will both be twice as big as in the second case, because the number of ways of permuting two cards is $P(2,2) = 2! = 2$. But when we take the ratio, this factor of two will cancel from the numerator and denominator. For simplicity, I will restrict my answer to the lower number of combinations.

So on to the answers. For convenience, let's denote the various aces by the following symbols: C = ace of clubs, D = ace of diamonds, H = ace of hearts, and S = ace of spades. In part (i), N_{win} is simply $C(4,2) = 6$, because there are 6 ways of getting two aces, namely {CD,CH,CS,DH,DS,HS}. On the other hand, $N_{lose} = C(48,2) + C(4,1)C(48,1) = 1320$ since $C(48,2)$ is the number of ways of getting two non-aces and $C(4,1)C(48,1)$ is the number of ways of getting one ace and one non-ace (where we multiply the two, using the fundamental principle of counting). Thus, the probability of any two aces is $6/(6+1320) = 1/221$. You can also get this answer more directly by simply multiplying the probability of getting one ace (4/52) by the probability of getting a second ace (3/51).

In part (ii), there are still 6 ways of getting any two aces. But this time, N_{lose} is only $C(4,1)C(48,1) = 192$ since the possibility of getting two non-aces is excluded. Thus, the probability of any two aces given one has risen to $6/(6+192) = 1/33$.

Finally, in part (iii), N_{win} is simply $C(3,1) = 3$, because if we already have the ace of spades, there are only 3 ways of getting another ace, namely {CS,DS,HS}. By the same reasoning, $N_{lose} = C(48,1) = 48$ since there are 48 ways of getting a non-ace instead. Thus, the probability of two aces given the ace of spades is $3/(3+48) = 1/17$. Again, you can get this more simply by noting that if you have an ace of spades, the probability of getting any other ace is 3/51.

As one might have expected, the more information you have, the more your odds go up. If you carefully examine the possibilities, you will realize that there are twice as many ways to get any two aces as to get the ace of spades plus any other ace. But there are four times as many ways to get any ace plus any non-ace as to get the ace of spades plus any non-ace. Thus, your odds roughly get halved if you only know you have an ace but don't know its suit.